POINTS OF INDETERMINATE SLOPE ON THE DISCRIMINANT LOCUS OF AN ORDINARY DIFFERENTIAL EQUATION.*

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The investigation of the solutions of an ordinary differential equation of the first order,

$$f(x, y, p) = 0, \quad \left(p = \frac{dy}{dx}\right), \tag{1}$$

depends primarily on the theory of implicit functions. If x_0, y_0, p_0 is a set of constants satisfying equation (1); if f is an analytic function of its three arguments in the region considered; and if the first partial derivative, f_p, of f as to p does not vanish for this set of values, we may apply the fundamental existence theorem on implicit functions and obtain a solution for p of the form

$$p = \varphi(x, y), \quad \text{where } p_0 = \varphi(x_0, y_0). \tag{2}$$

This solution is unique and φ can be expanded as a series in integral powers of $x - x_0$ and $y - y_0$. Cauchy's theorem is applicable to equation (2) and we know that through the point (x_0, y_0) in the direction determined by p_0 there passes one and only one integral curve of equation (1).

If the values x_0, y_0, p_0 satisfy not only equation (1) but also the equation

$$f_p(x, y, p) = 0, \tag{3}$$

the fundamental theorem on implicit functions can not be applied directly, and another method is required. If p can be eliminated between equations (1) and (3) the result is the discriminant equation

$$\Delta(x, y) = 0. \tag{4}$$

For any point (x_0, y_0) on the discriminant locus there is at least one value p_0 such that equations (1) and (3) are both satisfied. Consequently the fundamental theorem on implicit functions does not suffice for the immediate determination of all the integral curves through a point on the discriminant locus. In order to make this determination it has been necessary to place limitations on the form of the function f. Supposing f to be a rational integral function of y and p, Hamburger† has based his

* Read before the American Mathematical Society, December 28, 1910.

† Crelle's Journal, vol. 112 (1893), pp. 205–246. See also Schlesinger, Differentialgleichungen, Sammlung Schubert, Zweite Auflage, pp. 238–281, and Horn, Differentialgleichungen, S.S., pp. 348–356.

investigations on the known theory of algebraic functions of two variables, and, consequently, has excluded from consideration any point (x_0, y_0) for which equation (1) is satisfied identically in p. The principal object of this paper is an investigation of the integral curves through such a singular point. The first part is devoted to a consideration of equations of the second degree in p. In the latter part an extension is made under certain conditions to equations which are of higher degree, or even transcendental, in p.

It is convenient to recall here, for reference later, a result due to Briot and Bouquet.* They have studied an equation of the form $xp = \varphi(x, y)$, where φ can be expanded as a power series in x and y, and $\varphi(0, 0) = 0$. The term of the first power in y being of prime importance, the equation is written

$$xp - ay = a_{10}x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots. \tag{5}$$

Assuming a solution of the form

$$y = b_1 x + b_2 x^2 + \cdots, \tag{6}$$

the following results are obtained.

i. If a is not a positive integer, the coefficients b_i are uniquely determined, and there is one and only one solution of the type (6), that is representable as a series in integral powers of x.

ii. If $a = 1$, $a_{10} \neq 0$, there is no solution of the type (6).

iii. If $a = 1$, $a_{10} = 0$, the coefficient b_1 is arbitrary, while the remaining coefficients are uniquely determined in terms of the first one. A singly infinite number of integral curves with arbitrary slope pass through the origin.

iv. If $a = q$, where q is a positive integer greater than unity, it follows directly by substituting the value of y from equation (6) in equation (5) that the coefficients $b_1, \cdots b_{q-1}$ are uniquely determined. The equation for the determination of b_q either leads to a contradiction, that is, no solution of the type (6) exists, or is satisfied for an arbitrary value of b_q. In the latter case the remaining coefficients are uniquely determined in terms of b_q. A singly infinite number of integral curves all having the same slope pass through the origin.

I.

Consider the equation

$$f(x, y, p) \equiv A(x, y)p^2 + 2B(x, y)p + C(x, y) = 0. \tag{7}$$

In the region considered the coefficients A, B, and C are supposed to be

* Journal de l'école polytechnique, vol. 21 (1856), p. 161.

analytic functions of x and y, and equation (7) is supposed to be irreducible in x, y, and p. The discriminant equation is

$$\Delta(x, y) \equiv B^2 - AC = 0. \tag{8}$$

Let $y = \eta(x)$ be a solution of equation (8), where η is an analytic function of x in a certain neighborhood of the value $x = c$. We suppose that $A[x, \eta(x)] \not\equiv 0$, and, in particular, $A[c, \eta(c)] \neq 0$.*

From equation (7)

$$p = \frac{-B + \sqrt{\Delta}}{A}. \tag{9}$$

For any point not on the discriminant locus there are two distinct values of p. For any point on the curve $y = \eta(x)$ equation (7) has a double root

$$p = \zeta(x) = -\frac{B[x, \eta(x)]}{A[x, \eta(x)]}. \tag{10}$$

Now A, B, and C may be expanded as power series in $y - \eta$:

$$A(x, y) = A(x, \eta) + a_1(y - \eta) + \cdots,$$

$$B(x, y) = B(x, \eta) + b_1(y - \eta) + \cdots,$$

$$C(x, y) = C(x, \eta) + c_1(y - \eta) + \cdots,$$

where the coefficients are analytic functions of x. Since $\Delta[x, \eta(x)] \equiv 0$, the expansion for Δ has the form

$$\Delta(x, y) = (y - \eta)^{\alpha}[d_0 + d_1(y - \eta) + \cdots],$$

where $d_0(x) \not\equiv 0$, and we suppose, in particular, $d_0(c) \neq 0$.† Substituting these expansions in equation (9) gives

$$p = \frac{-B(x, \eta) - b_1(y - \eta) - \cdots + (y - \eta)^{\alpha/2}[d_0 + d_1(y - \eta) + \cdots]^{\frac{1}{2}}}{A(x, \eta)\left[1 + \dfrac{a_1}{A(x, \eta)}(y - \eta) + \cdots\right]}.$$

Since $A \neq 0$, $d_0 \neq 0$, the second member of this equation may be expanded in powers of $(y - \eta)^{\frac{1}{2}}$, giving

$$p - \zeta(x) = g_0(y - \eta)^{\frac{k}{2}} + g_1(y - \eta)^{\frac{k+1}{2}} + \cdots, \tag{11}$$

* If A vanishes for $x = c$, $y = \eta(c)$, while C does not vanish at this point, the problem is treated by considering the reciprocal of p.

† For a study of the case $d_0(c) = 0$ see a paper by Petrovitch, Mathematische Annalen, vol. 50 (1898), p. 103.

where k is a positive integer and the coefficients g_i are analytic functions of x in the neighborhood of $x = c$. There are now two cases to be considered.

Case I.—If $d\eta/dx \equiv \zeta$, then $y = \eta(x)$ is a solution of the given differential equation. In this case equation (11) may be written

$$\frac{d(y-\eta)}{dx} = g_0(y-\eta)^{\frac{k}{2}} + g_1(y-\eta)^{\frac{k+1}{2}} + \cdots.$$

By making the substitution $y - \eta = u^2$, this equation becomes

$$2u\frac{du}{dx} = g_0u^k + g_1u^{k+1} + \cdots. \tag{12}$$

The factor u may be canceled since $u = 0$ corresponds to the solution $y = \eta(x)$ already known. Cauchy's existence theorem is applicable to the remaining equation and asserts that there exists a unique solution for u vanishing with $x = c$.

If $k = 1$ this solution has the form

$$u = u_1(x-c) + u_2(x-c)^2 + \cdots,$$

where the coefficients u_i are not all zero. In this case we see that through the point $P[c, \eta(c)]$ besides the integral curve $S: y = \eta$, there passes one other integral curve

$$y = \eta(x) + u_1^2(x-c)^2 + \cdots.$$

This integral curve is tangent to S at the point P. Since P is any point on the curve S for which the hypotheses enumerated above are satisfied, it follows that S is an envelope of integral curves, that is, a *singular* solution.

If $k > 1$ the only solution of equation (12) is $u = 0$. In this case S is the only integral curve passing through P, and $y = \eta(x)$ is merely a *particular* solution.

Case II.—If $\frac{d\eta}{dx} \not\equiv \zeta$ we suppose, in particular, that

$$\zeta - \frac{d\eta}{dx} = \gamma \neq 0, \text{ for } x = c.$$

Equation (11) may be written

$$\frac{d(y-\eta)}{dx} = \zeta - \frac{d\eta}{dx} + g_0(y-\eta)^{\frac{k}{2}} + \cdots.$$

By making the substitution $y - \eta = u^2$ this equation becomes

$$2u\frac{du}{dx} = \zeta - \frac{d\eta}{dx} + g_0u^k + \cdots,$$

whence

$$\frac{dx}{du} = \frac{2u}{\zeta - \frac{d\eta}{dx} + g_0 u^k + \cdots}.$$

Since the denominator does not vanish for $x = c$, $u = 0$, the second member may be expanded in powers of $x - c$ and u, containing u as a factor:

$$\frac{dx}{du} = \frac{2}{\gamma} u + uP, \tag{13}$$

where P is a power series vanishing with $x = c$, $u = 0$. Equation (13) admits a unique solution for $x - c$ as a power series in u, and this series contains u^2 as a factor:

$$x - c = \frac{1}{\gamma} u^2 + \cdots.$$

Reverting this series we get

$$u = \sqrt{y - \eta} = \sqrt{\gamma}(x - c)^{\frac{1}{2}} + \cdots,$$

and, squaring, the solution of the original differential equation is found in the form

$$y - \eta = \gamma(x - c) + \gamma_1(x - c)^{\frac{3}{2}} + \cdots.$$

From this it appears that through the point $P[c, \eta(c)]$ there passes one integral curve of equation (7), and this curve has a cusp at P. Since $\gamma \neq 0$ the cuspidal tangent does not coincide with the tangent to the curve $y = \eta(x)$. In this case the branch $y = \eta$ of the discriminant curve is a locus of cusps on integral curves.

The conclusions reached so far may be summarized as follows: If Γ: $y = \eta(x)$ is an analytic branch of the discriminant locus three cases may occur. (1) $y = \eta(x)$ may be a particular solution. Through a general point on Γ there passes *no* other integral curve. (2) $y = \eta(x)$ may be a singular solution. Through a general point P on Γ there passes *one* other integral curve and it is tangent to Γ at P. (3) $y = \eta(x)$ may be a cusp locus. Through a general point P on Γ there passes *one* integral curve and it has a cusp at P. In general the cuspidal tangent does not coincide with the tangent to Γ at P.

The preceding results are due to Hamburger. Certain points have been excluded by the hypotheses made during the argument, in particular, those points for which A, B, and C all vanish. Suppose now that for a certain point, which will be taken for the origin, A, B, and C vanish. Let α be the degree of the terms of lowest degree in A, B, or C. Then

$$A(x, y) = a_{\alpha, 0}x^\alpha + a_{\alpha-1, 1}x^{\alpha-1}y + \cdots + a_{0, \alpha}y^\alpha + a_{\alpha+1, 0}x^{\alpha+1} + \cdots,$$
$$B(x, y) = b_{\alpha, 0}x^\alpha + \cdots, \quad C(x, y) = c_{\alpha, 0}x^\alpha + \cdots.$$

By making the substitution

$$(14) \qquad y = xv, \quad p = xv' + v, \quad \left(v' = \frac{dv}{dx}\right)$$

in equation (7), and canceling the factor x^α, we get

$$(15) \qquad A_1(xv' + v)^2 + 2B_1(xv' + v) + C_1 = 0,$$

where

$$A_1(x, v) = a_{\alpha, 0} + a_{\alpha-1, 0}v + \cdots + a_{0, \alpha}v^\alpha + a_{\alpha+1, 0}x + \cdots,$$
$$B_1(x, v) = b_{\alpha, 0} + \cdots, \quad C_1(x, v) = c_{\alpha, 0} + \cdots,$$

are power series in x and v which converge for all values of v, if x is small enough.

A first necessary condition for an analytic integral curve through the origin is that for $x = 0$, v satisfies the equation

$$(16) \qquad A_1(0, v)v^2 + 2B_1(0, v)v + C_1(0, v) = 0.$$

This is an algebraic equation of degree $\alpha + 2$ if $a_{0, \alpha} \neq 0$. The initial value of v is the slope of the integral curve at the origin. Hence there are in general $\alpha + 2$ critical slopes which the integral curve may conceivably have. From equation (15) we get

$$(17) \qquad xv' = \frac{-A_1v - B_1 + \sqrt{\Delta_1}}{A_1},$$

where $\Delta_1 = B_1^2 - A_1C_1$. The following cases are presented.

Case I.—A critical slope v_1 not tangent to a branch of the discriminant locus. In this case

$$B_1^2(0, v_1) - A_1(0, v_1)\, C_1(0, v_1) \neq 0,$$

and, if $A_1(0, v_1) \neq 0$, the second member of equation (17) may be expanded in powers of x and $v - v_1$, and the equation is of the form (5) investigated by Briot and Bouquet. There may be no analytic solution or there may be one of the form

$$v - v_1 = b_1x + b_2x^2 + \cdots,$$

where the coefficients are either uniquely determined or contain an arbitrary constant.

The solution of equation (7) becomes

$$y = v_1x + b_1x^2 + \cdots.$$

Hence through the singular point in the direction determined by v_1 there

may be either (1) no analytic integral curve, or (2) one, or (3) a singly infinite number.

Case II.—A critical slope v_1 tangent to an analytic branch of the discriminant curve at the origin. The equation of the branch of the discriminant curve may be written in the form $y = \eta(x) = x\psi$, where ψ is a power series in x. From equation (10)

$$p = \zeta(x) = -\frac{B_1(x, \psi)}{A_1(x, \psi)}.$$

For a singular or particular solution

(18) $$x\psi'(x) + \psi(x) \equiv \zeta(x).$$

For a cusp locus

(19) $$x\psi'(x) + \psi(x) \not\equiv \zeta(x),$$

but we might have

(20) $$\psi(0) = \zeta(0).$$

In this case Δ_1 vanishes for $x = 0$, $v = v_1$ and, if $A_1(0, v_1) \neq 0$, the second member of equation (17) can be expanded in powers of $(v - \psi)^{\frac{1}{2}}$ with coefficients which are analytic functions of x, and the result can be written

$$x\frac{d(v - \psi)}{dx} = -\psi - x\psi' + \zeta + g_0(v-\psi)^{\frac{k}{2}} + g_1(v - \psi)^{\frac{k+1}{2}} + \cdots.$$

By putting $v - \psi = u^2$ this equation becomes

(21) $$2xu\frac{du}{dx} = -\psi - x\psi' + \zeta + g_0u^k + g_1u^{k+1} + \cdots.$$

(*a*) If $y = x\psi(x)$ is a solution of equation (7) then the relation (18) shows that a factor u may be canceled from equation (21), leaving the equation of Briot and Bouquet. There may be no analytic solution or there may be one of the form

$$u = b_1x + b_2x^2 + \cdots,$$

where the coefficients are either uniquely determined or contain an arbitrary constant. The corresponding solution of equation (7) is

$$y = x\psi(x) + b_1^2x^3 + \cdots.$$

Hence if $y = x\psi(x)$ is a singular or particular solution of equation (7) there may be either (1) no other integral curve tangent to it at the singular point, or (2) one, or (3) a singly infinite number. (*b*) If $y = x\psi(x)$ is a cusp locus then relation (19) shows that a factor u can not be canceled from equation (21). In general the second member does not vanish for $x = 0$, $u = 0$, and there is no analytic solution. However, if the relation (20) holds, analytic solutions may exist. The equation is similar to equation (5), and may have

one or an infinite number of solutions. Hence if $y = x\psi(x)$ is a cusp locus, there is, in general, no integral curve tangent to it at the singular point; but there may be one (see example IV), or even an infinite number (see example V).

Case III.—A critical slope tangent to a non-analytic branch of the discriminant locus. This case is not treated in the present paper.

Case IV.—It may happen that equation (16) is satisfied identically in v, and hence the initial slope is arbitrary. Then a factor x may be canceled from both sides of equation (17) and Cauchy's theorem is applicable to the resulting equation for all initial values of v except a certain finite number for which $A_1 = 0$ or $\Delta_1 = 0$. Hence there are an infinite number of integral curves passing through the singular point, the slope being arbitrary, except for a finite number of directions which require special investigation. See examples I*b* and IV.

Example I.

$$\begin{aligned} f(x, y, p) \equiv x^3(1+x)^2p^2 + 2xy(1+x)[2y - x(3+4x)]p \\ + y^2[x(3+4x)^2 - 4y(2+3x)] = 0, \end{aligned} \tag{22}$$

$$\Delta(x, y) = 4x^2(1+x)^2y^3(y - x - x^2) = 0.$$

There are two analytic branches:

$$y = \eta(x) = 0, \text{ a particular solution,}$$

$$y = \eta(x) = x + x^2, \text{ a singular solution.}$$

(*a*) Equation (22) has a singular point at the origin, which is a point of intersection of the two analytic branches of the discriminant curve. For this example equation (17) becomes

$$xv' = \frac{v(2+3x) - 2v^2 + 2\sqrt{v^3[v-(1+x)]}}{1+x}. \tag{23}$$

There are two initial values of v, each falling under case II, namely, 0 and 1, and for each of these values the quantity under the radical vanishes.

Taking first the initial value $v = 0$, expanding the second member of equation (23) in powers of $v^{\frac{1}{2}}$ and setting $v = u^2$, we get (corresponding to equation (21))

$$2xu\frac{du}{dx} = 2u^2 + x(1+x)^{-1}u^2 + u^3P,$$

where P denotes a power series in x and u. Canceling the factor u which corresponds to the solution $y = 0$ already known, this equation becomes

$$x\frac{du}{dx} - u = \tfrac{1}{2}x(1 + x)^{-1}u + u^2P. \tag{24}$$

Equation (24) is of the form (5) and admits a solution

$$u = b_1x + b_2x^2 + \cdots,$$

where b_1 is arbitrary. The corresponding solution of equation (7) is

$$y = b_1^2x^3 + \cdots.$$

Hence an infinite number of integral curves pass through the singular point, all tangent to the particular solution $y = 0$.

Taking next the initial value $v = 1$, expanding the second member of equation (23) in powers of $(v - \psi)^{\frac{1}{2}}[\psi = 1 + x]$, and setting $v - \psi = u^2$, we get

$$2xu\frac{du}{dx} = 2u + uQ,$$

where Q denotes a power series in x and u, vanishing with x and u. Canceling the factor u which corresponds to the solution $y = x + x^2$ already known, it is seen that the second member does not vanish with x and u. Hence there exists no other analytic solution, that is, through this point there passes no integral curve tangent to the singular solution.

(*b*) Equation (22) has a singular point at $(-1, 0)$. Translating the origin to this point the equation becomes

$$\begin{aligned} f(x, y, p) \equiv x^2(x-1)^3p^2 &+ 2xy(x-1)[2y-(x-1)(4x-1)]p \\ &+ y^2[(x-1)(4x-1)^2 - 4y(3x-1)] = 0. \end{aligned} \tag{25}$$

The discriminant equation becomes

$$\Delta(x, y) = 4x^2(x-1)^2y^3[y + x - x^2] = 0.$$

There are two analytic branches passing through the origin:

(*a*) $y = x\psi(x) = 0$, a particular solution,

(*b*) $y = x\psi(x) = x(-1 + x)$, a singular solution.

For this example equation (17) becomes

$$xv' = \frac{3x(x-1)v - 2xv^2 + 2xv\sqrt{v^2 - v(x-1)}}{(x-1)^2}. \tag{26}$$

The initial value of v is arbitrary. When $x = 0$ the denominator does not vanish for any value of v, but the quantity under the radical vanishes for $v = 0$, $v = -1$. Canceling the factor x, the second member of equation (26) may be expanded in powers of x and $v - v_1$ where v_1 is arbitrary

except $v_1 \neq 0$, $v_1 \neq -1$. Cauchy's theorem is applicable to this equation and there exists a unique solution

$$v - v_1 = b_1 x + b_2 x^2 + \cdots.$$

The corresponding solution of equation (25) is

$$y = v_1 x + b_1 x^2 + \cdots.$$

Hence in every direction not tangent to a branch of the discriminant curve there passes *one* integral curve.

To examine the initial value $v = 0$ we expand the second member of equation (26) in powers of x and $v^{\frac{1}{2}}$. After making the substitution $v = u^2$ a factor u, corresponding to the solution $y = 0$ already known, may be canceled, and the resulting equation is

$$\frac{du}{dx} = uP,$$

where P denotes a power series in x and u. By Cauchy's theorem this equation admits a unique solution which is evidently $u = 0$. Hence $y = 0$ is the only integral curve through this point with slope equal to zero.

To examine the initial value $v = -1$ we expand the second member of equation (26) in powers of $(v - \psi)^{\frac{1}{2}}$, $[\psi = -1 + x]$, with coefficients which are analytic functions of x. After making the substitution $v - \psi = u^2$, a factor u, corresponding to the solution $y = -x + x^2$ already known, may be canceled, and the resulting equation is

$$\frac{du}{dx} = \frac{1}{\sqrt{x-1}} + uP.$$

By Cauchy's theorem this equation admits a unique solution

$$u = b_1 x + b_2 x^2 + \cdots.$$

The corresponding solution of equation (25) is

$$y = -x + x^2 + b_1^2 x^3 + \cdots.$$

Hence there is *one* other integral curve tangent to the singular solution at this point.

The results concerning equation (22) may now be summarized as follows.* Through any point not on the discriminant locus there are *two* integral curves with distinct tangents. Through any point on the

* It is understood that the direction of the Y-axis is tacitly excluded. Also the branches $x = 0$, $x = -1$ of the discriminant locus are excluded. They can not be represented by equations of the form $y = \eta(x)$.

branch $y = 0$ of the discriminant locus, except the singular points $O(0, 0)$ and $P(-1, 0)$, there passes only *one* integral curve, namely $y = 0$. Through any point on the branch $y = x + x^2$ of the discriminant locus there are two integral curves having the same direction. Through the singular point O there is one integral curve (the singular solution) with slope equal to unity, and an infinite number with slope equal to zero. Through the singular point P, in addition to the singular integral, there is *one* integral curve in every direction. The general solution of equation (22) is

$$y = c^2x^3\frac{1 + x}{2cx - 1}.$$

Example II.

(27) $$f(x, y, p) = Ap^2 + 2Bp + C = 0,$$

where

$$A = (x + y)^3 + (x - y)(2x + y)^2,$$

$$B = (x + y)^3 - (x - y)(2x + y)(x + 2y),$$

$$C = (x + y)^3 + (x - y)(x + 2y)^2.$$

$$\Delta(x, y) = 9(y + x)^5(y - x).$$

Of the two analytic branches of the discriminant locus, $y = -x$ is a particular solution and $y = x$ is a cusp locus. The origin is the only singular point. At any other point on the cusp locus the cuspidal tangent is perpendicular to the line $y = x$. For this example equation (17) becomes

(28) $$xv' = \frac{1 + v}{5 + 3v}\{4 - 3(1 + v)^2 + 3\sqrt{(1 + v)^3(v - 1)}\}.$$

There are three initial values of v, namely,

$$v_1 = \tfrac{1}{3}\sqrt{-3}, \quad v_2 = -\tfrac{1}{3}\sqrt{-3}, \quad v_3 = -1.$$

Expanding the second member in powers of $v - v_1$, equation (28) becomes

$$x\frac{d(v - v_1)}{dx} = g_0(v - v_1) + (v - v_1)^2P,$$

where

$$g_0 = \frac{(\sqrt{-3} - 9)(\sqrt{3} + \sqrt{-1})}{2\sqrt{3}(5 + \sqrt{-3})}.$$

This is the equation of Briot and Bouquet admitting a unique solution which is seen to be $v - v_1 = 0$. The corresponding solution of equation (27) is $y = \frac{1}{3}\sqrt{-3}x$. Expanding in powers of $v - v_2$ it is shown in a similar manner that the only solution is $v - v_2 = 0$, and the corresponding solution

of equation (27) is $y = -\frac{1}{3}\sqrt{-3}x$. These two integrals can be combined into a single one and written in the form $3y^2 + x^2 = 0$.

For the initial value $v_3 = -1$ the second member must be expanded in powers of $(v+1)^{\frac{1}{2}}$, and equation (28) becomes

$$(29) \qquad x\frac{d(v+1)}{dx} = 2(v+1) + (v+1)^{\frac{3}{2}}P,$$

where P denotes a power series in $(v+1)^{\frac{1}{2}}$. By setting $v + 1 = u^2$ equation (29) takes the form

$$xu\frac{du}{dx} = u^2 + \tfrac{1}{2}u^3P.$$

Canceling the factor u there is left the equation of Briot and Bouquet admitting a solution of the form

$$u = b_1x + b_2x^2 + \cdots,$$

where b_1 is arbitrary. The corresponding solution of equation (27) is

$$y = -x + b_1^2x^3 + \cdots.$$

Hence in addition to the curve $3y^2 + x^2 = 0$ which has a conjugate point at the origin, there are an infinite number of integral curves passing through this singular point each one of which is perpendicular to the cusp locus. There is no integral curve having a cusp at the origin.

The general integral of equation (27) is

$$(x+y)(x+y-c)^2 + (x-y)^3 = 0.$$

III.

Consider now an equation of more general form

$$(30) \qquad f(x, y, p) = 0.$$

To take up only the simplest case suppose that f can be expanded as a power series in $(x - a)$, $(y - b)$, and $(p - c)$, and that the following conditions hold.

$$(31) \qquad f(a, b, c) = 0,\ f_p(a, b, c) = 0,\ f_y(a, b, c) \neq 0,\ f_{pp}(a, b, c) \neq 0.$$

With this hypothesis the fundamental theorem on implicit functions may be applied to solve the equations $f = 0$, $f_p = 0$ for

$$(32) \qquad y = \eta(x), \quad \text{where } b = \eta(a),$$

$$(33) \qquad p = \zeta(x), \quad \text{where } c = \zeta(a).$$

Equation (32) is the discriminant locus and equation (33) gives the value of p (near $p = c$) at any point on this locus. For every value of x within a certain neighborhood of $x = a$ the following conditions are satisfied,

$$f[x, \eta(x), \zeta(x)] = 0, \quad f_p[x, \eta(x), \zeta(x)] = 0,$$

$$f_y[x, \eta(x), \zeta(x)] \neq 0, \quad f_{pp}[x, \eta(x), \zeta(x)] \neq 0.$$

Hence equation (30) may be written in the form

$$f = 0 = f_{10}(y - \eta) + f_{20}(y - \eta)^2 + f_{11}(y - \eta)(p - \zeta) + f_{02}(p - \zeta)^2 + \cdots,$$

where the coefficients are analytic functions of x. Since f_{10} and f_{02} do not vanish in the region considered this equation may be solved for $p - \zeta$ as a series in $(y - \eta)^{\frac{1}{2}}$:

$$p - \zeta = g_0(y - \eta)^{\frac{1}{2}} + g_1(y - \eta) + \cdots.$$

This equation is of the form (11) where $k = 1$. Therefore, if

$$\frac{d\eta}{dx} \equiv \zeta, \quad y = \eta(x)$$

is a singular solution; and, if

$$\frac{d\eta}{dx} \not\equiv \zeta, \quad y = \eta(x)$$

is a cusp locus.

Suppose now that for $x = 0$, $y = 0$ equation (30) is satisfied identically in p. Then it may be written in the form

$$f(x, y, p) = A(x, y) + B(x, y)p + \cdots = 0,$$

where, as before,

$$A(x, y) = a_{\alpha,0}x^\alpha + \cdots + a_{0,\alpha}y^\alpha + a_{\alpha+1,0}x^{\alpha+1} + \cdots, \quad B(x, y) = b_{\alpha,0}x^\alpha + \cdots.$$

Making the substitution (14) and canceling a factor x^α, gives

$$A_1(x, v) + B_1(x, v)\,(xv' + v) + \cdots = 0. \tag{34}$$

The initial values of v are determined by the condition

$$A_1(0, v) + B_1(0, v)v + \cdots = 0. \tag{35}$$

Suppose $v = c$ satisfies condition (35). Then equation (34) may be written in the form

$$A_2(x, v - c) + B_2(x, v - c)(xv' + v - c) + \cdots = 0, \tag{36}$$

where A_2, B_2, $\cdots$ are power series in x and $v - c$, and A_2 contains no constant term. If the initial slope $v = c$ is not tangent to a branch of the discriminant locus, that is, if B_2 does not vanish for $x = 0$, $v = c$, the fundamental theorem on implicit functions is applicable to equation (36) which may be solved for the quantity $xv' + v - c$ as a power series in x and $v - c$. Hence we are led to the equation of Briot and Bouquet with its known results. If the initial slope $v = c$ is tangent to a branch of the discriminant locus, the fundamental theorem on implicit functions gives no information concerning the solution. In special cases it is possible to solve equation (36) for $xv' + v - c$ as a series involving fractional powers, and the results may be investigated by the methods used in the preceding examples.

Example III.

$$f(x, y, p) = x\sqrt{1 - p^2} - y \arccos p = 0. \tag{37}$$

$$\Delta(x, y) = \frac{y}{x} - \cos\frac{\sqrt{x^2 - y^2}}{y} = 0.$$

The discriminant locus consists of an infinite number of straight lines passing through the origin, namely, of all the tangent lines which can be drawn from the origin to the curve $y = \sin x$. Any one of the branches is given by

$$y = x \cos m,$$

where m is a constant satisfying the equation

$$m - \tan m = 0. \tag{38}$$

The origin is a singular point to be investigated later. If m is any value, except 0, which satisfies equation (38), then conditions (31) are satisfied for

$$a \neq 0,\ b = a \cos m,\ c = \cos m.$$

Also, corresponding to equations (32) and (33),

$$y = \eta(x) = x \cos m, \quad p = \zeta(x) = \cos m.$$

Hence $y = x \cos m$ is a singular solution.

To investigate the singular point at the origin we have, corresponding to equation (35),

$$\sqrt{1 - p^2} - v \arccos p = 0. \tag{39}$$

This determines an infinite number of initial values v_1, namely, $v_1 = \cos m$, where m satisfies equation (38). The value $m = 0$ ($v_1 = 1$) requires special investigation. For any other initial value equation (39) may be solved for p as a series in $(v - v_1)^{\frac{1}{2}}$. The result is

$$x\frac{d(v-v_1)}{dx} = \sqrt{-2m\sin m}(v-v_1)^{\frac{1}{2}} + \cdots.$$

By putting $v - v_1 = u^2$ it is apparent that this equation admits no analytic solution (except $v - v_1 = 0$).

To investigate the initial value $v_1 = 1$ we expand arc cos p in powers of $(1-p)^{\frac{1}{2}}$.

$$\text{arc cos } p = \sqrt{2}\sqrt{1-p}R,$$

where

$$R = 1 + \tfrac{1}{12}(1-p) + \cdots$$

is a power series in $1 - p$. Hence the factor $\sqrt{1-p}$ may* be canceled from equation (39), and the remaining equation can be solved for $p - 1$ as a series in $v - 1$. We have to consider then

$$(40) \qquad x\frac{d(v-1)}{dx} - 2(v-1) = (v-1)^2P,$$

where P is a series in $v - 1$. This is the equation of Briot and Bouquet admitting a solution of the form

$$v - 1 = b_2x^2 + b_3x^3 + \cdots,$$

where b_2 is arbitrary. The corresponding solution of equation (37) is

$$y = x + b_2x^3 + b_3x^4 + \cdots.$$

Hence an infinite number of integral curves pass through the origin tangent to the branch $y = x$ of the discriminant locus. Through any other point on this line it is evident, by applying Cauchy's theorem to equation (40), that there is no other integral curve. There is no integral curve through the origin tangent to any other branch of the discriminant locus.

The general solution of equation (37) is

$$y = c\sin\frac{x}{c}.$$

Every curve of this family passes through the origin with slope equal to unity. If the branches of the discriminant locus are ordered according to increasing values of m, then, as x increases, every curve of the general integral touches each branch of the envelope in order.

Example IV.

$$x^4p^2 - 4xy^2p + 4y^3 = 0.$$

* Corresponding to this factor there is the solution $y = x$ of equation (37). This equation has the property of a reducible algebraic equation. It is apparent by inspection that $y = x + constant$ is an integral.

Example V.

$$18(x - y)(xp - y)^2 + (x + y)^5(p + 1)^2 = 0.$$

Example VI.

$$4(x - y)[(4x - y)p + x - 4y]^2 + (x + y)^7(p + 1)^2 = 0.$$

Example VII.

$$x\sqrt{1 - p^2} + y \text{ arc sin } p = 0.$$

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